

AD-R126 865

THE BOUNDED P-POINT CLASSES IN ROBUST HYPOTHESIS  
TESTING AND FILTERING(U) MOORE SCHOOL OF ELECTRICAL  
ENGINEERING PHILADELPHIA PA S A KASSAM OCT 82

1/1

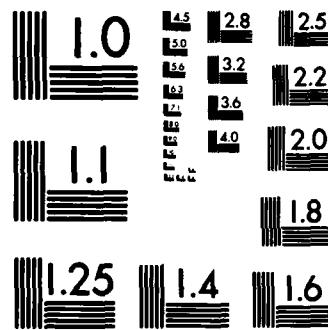
UNCLASSIFIED

AFOSR-TR-83-0286 AFOSR-82-0022

F/G 12/1

NL





MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS-1963-A

## UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER <b>AFOSR-TR- 33-0206</b>	2. GOVT ACCESSION NO. <i>AD-4126 FG5</i>	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle)  THE BOUNDED p-POINT CLASSES IN ROBUST HYPOTHESIS TESTING AND FILTERING	5. TYPE OF REPORT & PERIOD COVERED  Interim	
7. AUTHOR(s)  S. A. Kassam	6. PERFORMING ORG. REPORT NUMBER  AFOSR 82-0022	
9. PERFORMING ORGANIZATION NAME AND ADDRESS  Dept. of Systems Engineering, University of Pennsylvania Philadelphia, PA 19104	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS  61102F 2304/A5	
11. CONTROLLING OFFICE NAME AND ADDRESS  Air Force Office of Scientific Research Bolling AFB Washington, DC 20332	12. REPORT DATE  October, 1982	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	13. NUMBER OF PAGES  9	
	15. SECURITY CLASS. (of this report)  UNCLASSIFIED	
	15a. DECLASSIFICATION DOWNGRADING SCHEDULE	
16. DISTRIBUTION STATEMENT (of this Report)  Approved for Public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES  20th Annual Allerton Conf. on Comm., Control and Comp. October, 1982.		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)  Robust Wiener Filters, Robust Matched Filters, Robust Hypothesis Testing, Bounded Classes.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)  The bounded p-point classes of density functions are defined through upper and lower bounds on the power or probabilities on different sets in some parti- tion of the frequency or sample space. The robust Wiener filter is derived for such classes, and is shown to be piecewise constant in its frequency response. Similar results are indicated for hypothesis testing and matched filtering in- volving such classes of densities.		

FILE COPY

AD A126865

DTIC  
SELECTED  
APR 14 1983

H

SALEEM A. KASSAM

Department of Systems Engineering  
Moore School of Electrical Engineering  
University of Pennsylvania  
Philadelphia, PA. 19104

ABSTRACT

The bounded p-point classes of density functions are defined through upper and lower bounds on the power or probabilities on different sets in some partition of the frequency or sample space. The robust Wiener filter is derived for signal and noise spectra lying in such classes, and is shown to be piecewise constant in its frequency response. Similar results are indicated for hypothesis testing involving such classes of probability densities. The robust matched filter is also obtained for the bounded p-point class of noise spectra, and is again shown to have a piecewise-constant frequency response.

I. INTRODUCTION

In recent years a considerable number of results has been obtained on minimax robust filters operating under conditions of inexact a priori knowledge of signal and noise characteristics. Specifically, filters for estimation, smoothing, prediction, and interpolation of random signals have been considered [e.g., 1-3], in addition to matched filters for signal-to-noise ratio optimization [e.g., 4], and other filters for special applications, e.g., time-delay estimation. In all these cases a priori uncertainties on signal and noise characteristics are modeled by classes of possible second-order characteristics, i.e., spectral densities or covariance functions. Many of these results were motivated by earlier work on robust hypothesis testing [5], in which minimax robust tests for binary hypotheses were obtained for some specific classes of probability density functions. In particular, minimax robust solutions for robust Wiener filtering under spectral uncertainty classes and for robust hypothesis testing under corresponding probability density function classes are very closely related.

Although some rather general results are available on robust hypothesis testing and robust Wiener filtering under general uncertainty classes, solutions for useful specific classes are of particular interest in applications. The spectral and probability density models for which specific solutions have been obtained include the following: 1) the  $\epsilon$ -contaminated model [5]; 2) the "band-model" for bounded densities [6]; 3) the bounded total-variation model [5]; and 4) the specified p-point model [7].

Of these, perhaps of primary importance in applications are classes generated by the "band-models" and the specified p-point models. In the band-classes density functions are assumed to lie between upper and lower bounding functions, and in the specified p-point classes density functions are assumed to have specified proportions of their total area in specified regions in the sample-space or frequency domain. A natural and useful synthesis of these two types of classes yields the bounded p-point class of densities, where density functions are assumed to have proportions of their total area in specified regions lying between specified upper and lower bounds. It is easily seen that the bounded p-point classes include as

\*This research is supported by the Air Force Office of Scientific Research under Grant AFOSR 82-0022.

PRESENTED AT THE 20th ANNUAL ALLERTON CONFERENCE ON COMMUNICATION, CONTROL, AND COMPUTING OCT. 1982. TO BE PUBLISHED IN THE CONFERENCE PROCEEDINGS.

88 04 14 017

Approved for public release; distribution unlimited.

a limiting case the band-classes, and as a special case the specified p-point classes.

We obtain the structure of robust Wiener filters for signal and noise spectra in bounded p-point classes defined by a common partition of the frequency space, by using general results on robust hypothesis testing for bounded classes [6]. The problem is first converted to one involving discrete spectra, and then results on robust hypothesis testing for discrete probability distributions are used. The robust Wiener filter has a piecewise constant frequency response. In a similar way it can be shown that robust hypothesis testing for bounded p-point classes of density functions requires the use of a quantizer operating on input data. For Wiener filtering we also obtain a result for the worst-case performance of any filter when signal and noise spectra lie in bounded p-point classes or the bounded classes of the "band" model. The above results are obtained for signal and noise classes defined on the same partition of the sample or frequency space. We also show that a solution can be obtained when one of the densities is in the bounded class generated by the band-model.

For matched filtering under noise spectral density uncertainty, we also consider the bounded p-point class for the noise spectra. For known signal the solution is simple. When the signal is also uncertain, lying in the previously used bounded  $L_2$ -deviation class [4], we can also obtain a general solution. The interesting result here is that the robust matched filter frequency response is again a piecewise-constant function.

## II. ROBUST HYPOTHESIS TESTING AND WIENER FILTERING

We start by seeking a solution for the robust Wiener filter when signal and noise power spectral densities (PSD's) are members of bounded p-point classes. This will involve the robust binary hypothesis test for discrete "band" classes of probability densities. We define the bounded p-point class  $S$  of allowable signal PSD's  $S$  by

$$S = \{S: \int_{A_j} S(f) df = \sigma_j^2, j = 1, 2, \dots, m, \sum_{j=1}^m \sigma_j^2 = \sigma^2\} \quad (1a)$$

Here  $\{A_j\}_{j=1}^m$  is a given partition of the frequency space (with  $f \in A_j \rightarrow -f \in A_j$ ), and  $\sigma^2$  is the known total signal power, with the fractional powers  $\sigma_j^2$  satisfying

$$\sigma_{Lj}^2 \leq \sigma_j^2 \leq \sigma_{Uj}^2, j = 1, 2, \dots, m \quad (1b)$$

where the bounds  $\sigma_{Lj}^2, \sigma_{Uj}^2$  are known. In a similar way we define the class

$N$  of allowable noise PSD's  $N$  by

$$N = \{N: \int_{A_j} N(f) df = \nu_j^2, j = 1, 2, \dots, m, \sum_{j=1}^m \nu_j^2 = \nu^2\} \quad (2a)$$

and

$$\nu_{Lj}^2 \leq \nu_j^2 \leq \nu_{Uj}^2, j = 1, 2, \dots, m \quad (2b)$$

Note that we are assuming here a common partition  $\{A_j\}_{j=1}^m$  for both signal and noise classes.

Dist

We now consider a reduced discrete spectral distribution problem. Let

A

all signal, noise power in each subset  $A_j$  be concentrated at a single frequency pair  $f_j \in A_j$ . We will denote by  $S_d, N_d$  the resulting classes of one-sided power spectral concentrations derived from  $S, N$ , respectively. Let  $H$  be the frequency response of any filter used for estimating the signal component with power spectrum  $S_d = [S_d(1), S_d(2), \dots, S_d(m)]$  in  $S_d$ , observed in additive uncorrelated noise with power spectrum  $N_d \in N_d$ . Let  $H_d(j) = H(f_j)$ ,  $j = 1, 2, \dots, m$ . Then the mean-squared-error (mse)  $e_d(S_d, N_d; H_d)$  in estimating the signal for this discrete PSD's case is

$$e_d(S_d, N_d; H_d) = \sum_{j=1}^m S_d(j) |1 - H_d(j)|^2 + N_d(j) |H_d(j)|^2 \quad (3)$$

For given  $S_d, N_d$  this is minimized for  $H_d(j) = S_d(j) / [S_d(j) + N_d(j)]$  and the corresponding minimum mse is

$$e_d^*(S_d, N_d) = \sum_{j=1}^m S_d(j) N_d(j) / [S_d(j) + N_d(j)] \quad (4)$$

To obtain the least favorable pair  $(S_d^*, N_d^*)$ , which is defined to be the pair maximizing (4), we examine the corresponding hypothesis testing problem for discrete densities. Consider a random variable with a finite set of possible realizations  $\{1, 2, \dots, m\}$ , with a probability mass function given by either  $\{P_0(j)\}_{j=1}^m$  or  $\{P_1(j)\}_{j=1}^m$ . We assume that  $P_0$  and  $P_1$  are in the bounded classes defined by

$$P_0 = \{P_0 : P_{0L}(j) \leq P_0(j) \leq P_{0U}(j), j = 1, \dots, m\} \quad (5a)$$

$$P_1 = \{P_1 : P_{1L}(j) \leq P_1(j) \leq P_{1U}(j), j = 1, 2, \dots, m\} \quad (5b)$$

Now the pair of densities  $(P_0^*, P_1^*)$  in these classes which is least-favorable (risk) in testing  $P_0$  vs.  $P_1$  [6] can be found by applying the general result in [6] for "band" classes of bounded probability density functions. The results in [6] are applicable for both absolutely continuous and discrete probability distributions. Note that the definition of the pair  $(P_0^*, P_1^*)$  which is least-favorable (risk) implies that a corresponding optimum test for  $P_0^*$  vs.  $P_1^*$  is robust for testing  $P_0$  vs.  $P_1$ . On the other hand, the previous definition of the least-favorable pair  $(S_d^*, N_d^*)$  does not automatically imply robustness of the corresponding optimal filter, although for our case this will be shown to be true. In order to solve for the least-favorable pair  $(S_d^*, N_d^*)$  from the results in [6] applied to discrete probability distributions, we use the following lemma which is a direct counterpart of Lemma 1 in [2] where it is stated for the absolutely continuous case:

Lemma 1 Let  $P_0$  and  $P_1$  be classes of discrete probability mass functions  $P = \{P(j)\}_{j=1}^m$  such that each member of  $P_0 \cup P_1$  has positive values on the common set  $\{1, 2, \dots, m\}$ . If  $P_0^*$  and  $P_1^*$  are least-favorable (risk) for  $P_0$  vs.  $P_1$  then

$$\sum_{j=1}^m \Psi[P_1^*(j) / P_0^*(j)] P_0^*(j) \geq \sum_{j=1}^m \Psi[P_1(j) / P_0(j)] P_0(j) \quad (6)$$

for all  $P_0 \in P_0$ ,  $P_1 \in P_1$  and all continuous concave functions  $\Psi$ .

44

The proof of this lemma follows the same ideas as those used in the proof of lemma 1 in [2]. Although this lemma can be extended to apply to the slightly more general case, we will instead restrict our original classes of spectral densities by requiring that the lower bounds  $\sigma_{Uj}^2$  and  $\nu_{Lj}^2$  be positive (except in the case where both  $\sigma_{Uj}^2$  and  $\nu_{Lj}^2$  are zero). Now we transform the original classes  $S_d$  and  $N_d$  into classes  $P_s$  and  $P_n$  of probability mass functions  $P_s$  and  $P_n$  respectively, by scaling each member of  $S_d$  by  $1/\sigma^2$  and scaling each member of  $N_d$  by  $1/\nu^2$ . Thus, the  $P \in P$  are defined by  $P_s(j) = S_d(j)/\sigma^2$ ,  $j=1,2,\dots,m$ . Let  $(P_s^L, P_n^L)$  be the least-favorable (risk) pair in  $P_s \times P_n$ . Now  $e_d^*$  of (4) can be written as

$$e_d^*(P_s, P_n) = \sigma^2 \nu^2 \sum_{j=1}^m \frac{L(j)}{\sigma^2 L(j) + \nu^2} P_n(j) \quad (7)$$

where  $L(j) = P_s(j)/P_n(j)$ . The function  $x/(\sigma^2 x + \nu^2)$  is a concave function of  $x$ , and lemma 1 gives the result that  $S_d^L = \sigma^2 P_s^L$  and  $N_d^L = \nu^2 P_n^L$  form the least favorable pair maximizing (4) over the classes  $S_d$  and  $N_d$ .

Once the least favorable pair  $(S_d^L, N_d^L)$  has been characterized, it remains to be shown that the optimum filter for this signal, noise spectra is the robust filter minimizing the maximum mse over the classes  $S_d$  and  $N_d$ . This result is also a direct counterpart of Theorem 1 in [2] for the continuous case, and may be stated as follows:

Lemma 2 Let  $S_d$ ,  $N_d$  be convex classes of power spectral concentrations on a finite number of frequencies. Then  $(S_d^L, N_d^L) \in S_d \times N_d$  is least-favorable for Wiener filtering iff  $(S_d^L, N_d^L)$  and  $H_{d,R} = S_d^L / (S_d^L + N_d^L)$  form a saddlepoint for the mse functional  $e_d(S_d, N_d; H_d)$ .

Proof It is clear that if a pair  $(S_d^L, N_d^L)$  and its optimal filter  $H_{d,R}$  form a saddlepoint for mse, then the pair is least-favorable. To prove the converse we note, as in the proof in [2], that the functional  $e_d^*(S_d, N_d)$  is concave in  $S_d$  and in  $N_d$ , and that  $S_d^L, N_d^L$  are assumed to be convex. From this the proof proceeds in a manner similar to that in [2], except that now we use the mse expressions (3) and (4) in place of the error expressions for the continuous case.

We thus find that the saddlepoint robust filter for the bounded classes  $S_d$ ,  $N_d$  of discrete spectra can be found as the optimal filter for the least-favorable pair in  $S_d \times N_d$ . This least favorable pair is obtainable as a scaled version of the least-favorable (risk) pair for testing the hypothesis  $P_s$  vs.  $P_n$  where  $P_s$  and  $P_n$  are derived by scaling members of  $S_d$  and  $N_d$  respectively. We now finally obtain the saddlepoint robust filter  $H_R$  for the original bounded p-point spectral classes  $S$  and  $N$ .

Lemma 3 If  $H_{d,R}$  is the saddlepoint robust filter for the classes  $S_d, N_d$  of discrete spectra derived from the bounded p-point classes  $S, N$  defined by (1) and (2), then the filter defined by

$$H_R(f) = H_{d,R}(j), f \in A_j \quad (8)$$

is the saddlepoint robust filter for spectral classes  $S$  and  $N$ .

Proof We have, for  $S \in S$  and  $N \in N$ ,

$$e(S, N; H_R) = \int S(f) |1 - H_R(f)|^2 + N(f) |H_R(f)|^2 df \quad (9)$$

$$\begin{aligned}
&= \sum_{j=1}^m |1-H_{d,R}(j)|^2 \int_{A_j} S(f) df + \sum_{j=1}^m |H_{d,R}(j)|^2 \int_{A_j} N(f) df \\
&= \sum_{j=1}^m |1-H_{d,R}(j)|^2 S_d(j) + |H_{d,R}(j)|^2 N_d(j) \\
&\leq \sum_{j=1}^m |1-H_{d,R}(j)|^2 S_d^{\ell}(j) + |H_{d,R}(j)|^2 N_d^{\ell}(j) \\
&= e(S^{\ell}, N^{\ell}; H_R) \\
&= e^*(S^{\ell}, N^{\ell}),
\end{aligned}$$

where  $S^{\ell}, N^{\ell}$  is any PSD's in  $S, N$  respectively for which  $S^{\ell}(f)/N^{\ell}(f) = S_d^{\ell}(j)/N_d^{\ell}(j)$  when  $f \in A_j$ . Note that such a pair is least-favorable.

What we have obtained is an explicit solution for the piecewise-constant robust Wiener filter for bounded p-point classes of signal and noise spectra. In a similar way, we can obtain a corresponding result for robust hypothesis testing for bounded p-point classes of probability density functions.

Returning to the Wiener filtering problem, let  $S$  be a bounded p-point class of signal spectra, defined with respect to some partition  $\{A_j\}_{j=1}^m$ . Let  $\bar{N}$  be any class of noise spectra, and let  $\bar{N}$  be the class of bounded p-point spectra generated by  $N$ , for the same partition  $\{A_j\}_{j=1}^m$ . We then consider the least-favorable pairs  $(S^{\ell}, \bar{N}^{\ell})$  for these classes  $S$  and  $\bar{N}$  of signal and noise spectra. If a least-favorable pair exists with  $\bar{N}^{\ell} \in \bar{N}$ , then that pair is also least favorable for the original classes  $S$  and  $\bar{N}$ . We can apply this observation to the case where the class  $\bar{N}$  is a "band" class of bounded spectra as in [1]. From the proof of lemma 3 we note that  $S^{\ell}, \bar{N}^{\ell}$  are required to be constant multiples of each other, their exact shapes not being constrained. Therefore it is always possible to find a least-favorable pair  $(S^{\ell}, \bar{N}^{\ell})$  for classes  $S$  and  $\bar{N}$  such that  $\bar{N}^{\ell}$  is in the band class  $\bar{N}$ . This implies that the more precise bounds on noise spectra defining  $\bar{N}$  represents extra information, beyond bounds on fractional powers, which is not useful in obtaining a better robust filter. Note that in the above, the roles of  $S$  and  $N$  can be interchanged and similar statements can be made. For hypothesis testing, again similar conclusions can be drawn.

Before considering matched filters in the next section, we establish a result which allows computation of the worst-case performance of any filter for signal and noise spectra defined either by "band" models or the bounded p-point classes.

In general, for given signal and noise classes it is of interest to have the worst-case performance of any given filter  $H$ . Consider the power-constrained band classes of spectra for Wiener filtering. Note that the mse obtained with any filter frequency response  $H$  is given by

$$e(S, N; H) = \int N(f) |H(f)|^2 df + \int S(f) |1-H(f)|^2 df$$

Consider any integral of the form  $I = \int X(f) P(f) df$  where  $P(f)$  is given and  $X(f)$  lies in a power constrained "band" class of power spectra. Then the worst (maximum) value of  $I$  is obtained with  $X$  in the "band" class defined by

$$X_w(f) = \begin{cases} X_L(f), & P(f) < k \\ X_U(f), & P(f) > k \\ \text{arbitrary,} & P(f) = k \end{cases} \quad (10)$$

where  $k$  is a constant chosen to satisfy the power constraint. The proof follows easily by considering  $\int [X(f) - X_w(f)]P(f)df$  for any other  $X$ . This result can be applied to each of the two terms in the above error expression to obtain the worst case signal and noise spectra in the "band" class for any given filter. A direct extension gives the worst-case spectra for the bounded  $p$ -point classes.

### III. BOUNDED $p$ -POINT NOISE CLASS IN ROBUST MATCHED FILTERING

For noise characteristics described in the frequency domain, the frequency response of the robust matched filter was derived explicitly in [4] for noise spectral uncertainty modeled by the "band" classes of allowable spectra. The signals were required to be within a given  $L_2$  distance of a nominal signal. Here we will obtain results for the case where the noise class is the bounded  $p$ -point class described by (2). If  $S$  is the Fourier transform of the real deterministic signal and  $S_o$  is the nominal characteristic, the signal class  $S$  is now defined by

$$S = \{S: \int_{\Omega} |S(f) - S_o(f)|^2 df \leq \delta\} \quad (11)$$

Here  $\Omega$  is  $(-\infty, \infty)$  for the continuous case or  $[-0.5, 0.5]$  for the discrete case. The signal-to-noise ratio obtained when using a filter frequency response  $H$ , when  $S$  and  $N$  are the given characteristics, is given by

$$\text{SNR}(S, N; H) = \frac{\int |S(f)H(f)|^2 df}{\int |N(f)H(f)|^2 df} \quad (12)$$

where integrals are over  $\Omega$ . The following theorem gives our general result defining the least-favorable pair  $(S_R, N_R)$ , which together with the corresponding optimum filter  $H_R$  forms a saddlepoint for the SNR functional.

Theorem 1 For the bounded  $p$ -point noise class of (2) and the bounded  $L_2$ -deviation signal class of (11), the pair  $(S_R, N_R)$  defined below is least-favorable for matched filtering and the optimum filter  $H_R$  for  $(S_R, N_R)$  is a saddlepoint robust (matched) filter, provided non-negative constants  $k$  and  $c$  exist satisfying (17) and (18). The pair  $(S_R, N_R)$  is defined by

$$S_{Rj}(f) = \begin{cases} [ |S_o(f)| - k_j c ] e^{i \arg S_o(f)}, & |S_o(f)| \geq k_j c \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

and

$$N_{Rj}(f) = |S_{Rj}(f)| / k_j. \quad (14)$$

for  $j=1, 2, \dots, m$ , where  $S_R, N_R$  are the restrictions of  $S, N$  to the sets  $A_j$ , respectively. Here  $k_j$  are defined as functions of  $S_R, N_R, k$  and  $c$  through the equations

$$\sigma_{Rj}^2 = \int_{A_j} |S_{Rj}(f)|^2 df, \quad (15)$$

and

$$k_j = \begin{cases} \sigma_{Rj}^2 / v_{Uj}^2, & \sigma_{Rj}^2 > k v_{Uj}^2 \\ \sigma_{Rj}^2 / v_{Lj}^2, & \sigma_{Rj}^2 \leq k v_{Lj}^2 \\ k, & k v_{Lj}^2 < \sigma_{Rj}^2 \leq k v_{Uj}^2 \end{cases} \quad (16)$$

with the constraints

$$\sum_{j=1}^m \sigma_{Rj}^2 / k_j = v^2 \quad (17)$$

and

$$\sum_{j=1}^m \{c^2 k_j^2 \int_{R_j} df + \int_{A_j - R_j} |S_o(f)|^2 df\} = S, \quad (18)$$

the set  $R_j$ ,  $j = 1, 2, \dots, m$ , being defined as that subset of  $A_j$  on which  $|S_o(f)| > k_j c$ . In (14) for  $N_{Rj}(f)$  we set  $N_{Rj}(f)$  to be any function integrating to  $v_{Lj}^2$  on  $A_j$  if  $\sigma_{Rj}^2 = 0$ . The robust filter is

$$H_R(f) = \begin{cases} k_j \exp(-i \arg S_o(f)), & f \in R_j \\ S_o^*(f)/c, & f \in A_j - R_j. \end{cases} \quad (19)$$

Comments Note that (17) and (18) are the constraints on total noise power and on maximum deviation from the nominal signal for signals in  $S$ , respectively. The solution for  $S_R$  and  $N_R$  in any specific case requires a numerical procedure for evaluating the constants. In principle, this procedure can be an iterative procedure starting from some guess for  $k$  and  $c$ . For given  $k$  and  $c$  a solution for the  $k_j$  and  $\sigma_{Rj}^2$  can be sought from (15) and (16). Then the constraints (17) and (18) are tested. A proof is given in the Appendix.

It is also interesting to note that as the partition  $\{A_j\}_{j=1}^m$  becomes finer the solution approaches that for the "band" class for  $N_R$  in [4]. In the special case where only a noise power constraint is given ( $j=1$ ) the robust filter  $H_R = S_o^*/N_R = k_j \exp(-i \arg S_o(f))$  on  $R_j$  has a constant magnitude.

From (13) and (14), the optimum filter for  $S_R$ ,  $N_R$  is not uniquely defined where  $|S_o(f)| < k_j c$ . On this set the robust filter is simply  $S_o^*(f)/c$ , i.e., a filter matched to  $S_o$  for white noise.

#### IV. ROBUST TIME-DELAY ESTIMATION

In a recent paper [8], the specified  $p$ -point classes and the band classes of bounded spectral densities have been used in obtaining robust versions of the Eckart filter [8], which is a specific type of weighting function used in the computation of the cross-correlation function in time-delay estimation. The basic problem is to estimate the time delay  $D$  between noisy signals received at two sensors, the sensor inputs being described by  $v_1(t) = s(t) + n_1(t)$  and  $v_2(t) = s(t-D) + n_2(t)$ . Here  $s(t)$  is a stationary random signal with spectrum  $S$  and  $n_1$  and  $n_2$  are uncorrelated noise processes independent of  $s$ , with spectra  $N_1$  and  $N_2$ , respectively. The technique of time-delay estimation considered in [8] is based on forming an estimate of the cross-

spectral density  $G_{12}(f)$  of the inputs  $v_1$  and  $v_2$ , weighting this with a filter function  $W$ , and then taking the inverse transform to obtain an estimate of the cross-correlation function  $R_{12}(\tau)$ . The position of the peak in  $R_{12}(\tau)$  is taken as an estimate of the time delay  $D$ . For low signal-to-noise ratio at the input one appropriate performance measure is the output SNR of the estimate of  $R_{12}(D)$ . This is given by

$$d(S, Q; W) = \frac{[\int W(f)S(f)df]^2}{\int W^2(f)Q(f)df} \quad (20)$$

where  $Q(f) = N_1(f)N_2(f)$ . The spectral product  $Q$  and the signal spectrum can both be estimated from measurements at the system output. For given  $S$  and  $Q$  the optimum (Eckart) filter maximizing  $d(S, Q; W)$  is  $W_o = S/Q$ , and this maximum distance is  $d_o(S, Q) = \int S^2/Q df$ .

The results in [8] show that for convex classes of allowable  $S$  and  $Q$ , the least-favorable pair  $(S_R, Q_R)$  for  $(S, Q)$ , minimizing  $d(S, Q)$ , and the corresponding optimum filter  $W_R = S/Q_R$  form a saddle point for  $d(S, Q; W)$ . Thus, the saddlepoint robust filter can be determined once the least favorable pair  $(S_R, Q_R)$  is known. Now since  $d(S, Q)$  can be written as  $\int (S/Q)^2 Q df$ , lemma 1 can be used to obtain directly the least favorable pair from the corresponding hypothesis testing problem. For  $S$  and  $Q$  lying in power constrained bounded  $p$ -point classes the least favorable pair is exactly the same as for Wiener filtering in Section II, where  $Q$  plays the role of a noise spectrum.

## V. CONCLUSION

We have considered robust hypothesis testing and robust random signal estimation, deterministic signal detection and time-delay estimation, for cases where probability and spectral density functions belong to power-constrained bounded  $p$ -point classes. These classes are of practical interest because the bounds can be readily estimated in many situations. They form a generalization of the specified  $p$ -point classes and also of band-classes, which can be viewed as limiting cases of the bounded  $p$ -point classes.

## REFERENCES

- [1] S.A. Kassam and T.L. Lim, "Robust Wiener Filters," J. Franklin Inst., Vol. 304, pp. 171-185, 1977.
- [2] H.V. Poor, "On Robust Wiener Filtering," IEEE Trans. Auto. Control, Vol. AC-25, pp. 531-536, 1980.
- [3] K.S. Vastola and H.V. Poor, "Robust Wiener-Kolmogorov Theory," IEEE Trans. Inform. Theory, to be published.
- [4] S.A. Kassam, T.L. Lim and L.J. Cimini, "Two-dimensional Filters for Signal Processing Under Modeling Uncertainties," IEEE Trans. Geoscience and Remote Sensing, Vol. GRS-18, pp. 331-336, 1980.
- [5] P.J. Huber, "A Robust Version of the Probability Ratio Test," Ann. Math. Stat., Vol. 36, pp. 1753-1758, 1965.
- [6] S.A. Kassam, "Robust Hypothesis Testing for Bounded Classes of Probability Densities," IEEE Trans. Inform. Theory, Vol. IT-27, pp. 242-247, 1981.
- [7] L.J. Cimini and S.A. Kassam, "Robust and Quantized Wiener Filters for  $p$ -Point Spectral Classes," Proc. 14th Annual Conf. on Inform. Sciences and Systems, Princeton, pp. 314-319, 1980.
- [8] E.K. Al-Hussaini and S.A. Kassam, "Robust Filters for Time-Delay Estimation Problems," Proc. 16th Annual Conf. on Inform. Sciences and Systems, Princeton, pp. 540-545, 1982

APPENDIX: Proof of Theorem in Section III.

Consider the denominator  $D(N; H) = \int N |H|^2 df$  in (12). We have

$$D(N; H_R) - D(N_R; H_R) = \sum_j \left[ k_j^2 R_j(f) + \frac{|S_o(f)|^2}{c^2} \bar{R}_j(f) \right] (N - N_R) df, \quad (A1)$$

where  $R_j(f) = 1$  for  $f \in R_j$  and 0 otherwise, and where  $\bar{R}_j = A_j - R_j$ . Now on  $R_j$   $N_R$  is 0 and  $|S_o|^2/c^2 \leq k_j^2$ , so

$$\begin{aligned} D(N; H_R) - D(N_R; H_R) &\leq \sum_j k_j^2 \int [A_j(f)N - R_j(f)N_R] df \\ &\leq \sum_{k_j > k} k^2 \left[ \int A_j(f)N df - \int_{U_j}^2 \right] \\ &\quad + \sum_{k_j < k} k^2 \left[ \int A_j(f)N df - \int_{L_j}^2 \right] + \sum_{k_j = k} k^2 \left[ \int A_j(f)N df - \sigma_{R_j}^2/k \right] \end{aligned} \quad (A2)$$

The RHS of (A2) is 0, so  $D(N; H_R) \leq D(N_R; H_R)$  for all  $N \in N$ .

For the numerator, we first show that  $\int S H_R df \geq 0$ , all  $S \in S$ . This follows by noting that, as in (A1),

$$H_R(f) = \sum_j [k_j R_j(f) + \bar{R}_j(f) |S_o(f)|/c] \exp(-i \arg S_o(f)),$$

replacing  $S$  by  $S_o + (S - S_o)$ . The integral containing  $S_o$  only is independent of  $S$  and is lower bounded by  $\delta/c$ , from (18). The second integral [integrand containing  $(S - S_o)$ ] is real-valued and, from Schwarz's inequality, bounded by  $\delta/c$  in absolute value. Thus we consider minimization of this integral which is

$$\int \left[ \sum_j [k_j R_j(f) + \bar{R}_j(f) |S_o(f)|/c] |S - S_o| \cos[\arg(S - S_o) - \arg S_o] \right] df$$

To minimize the cos term is -1, when  $\arg(S - S_o)$  and  $\arg S_o$  have a difference of  $\pi$ . Without the cos term, the integral is maximized for  $S$  given by (13), from a direct application of Schwarz's inequality and the above observation on phase.

END

FILMED

5-83

DTIC